

Homework 13 Solns

1.

- ❶ Chapter 14, Section 7, Problem 56.

Find the inverse Laplace transform of the function $F(p) = \frac{1}{p^4-1}$ by using (7.16).

- ❷ Chapter 14, Section 7, Problem 62.

Find the inverse Laplace transform of the function $F(p) = \frac{(p-1)^2}{p(p+1)^2}$ by using (7.16).

Solution:

- ❶ We first find the poles of $F(z)e^{zt}$ and factor the denominator to get

$$F(z)e^{zt} = \frac{e^{zt}}{(z-1)(z+1)(z-i)(z+i)}.$$

Evaluating the residues at the four simple poles, we find

residue at $z = 1$	is	$\frac{e^t}{4}$
residue at $z = -1$	is	$-\frac{e^{-t}}{4}$
residue at $z = i$	is	$-\frac{e^{it}}{4i}$
residue at $z = -i$	is	$\frac{e^{-it}}{4i}$

Then by (7.16) we have

$$f(t) = \frac{e^t}{4} - \frac{e^{-t}}{4} - \frac{e^{it}}{4i} + \frac{e^{-it}}{4i} = \frac{\sinh t - \sin t}{2}.$$

- ❷ We first find the poles of $F(z)e^{zt}$ and factor the denominator to get

$$F(z)e^{zt} = \frac{(z-1)^2 e^{zt}}{z(z+1)^2}.$$

Evaluating the residues at the two poles, we find

residue at $z = 0$	is	1
residue at $z = -1$	is	$-4te^{-t}$

Then by (7.16) we have

$$f(t) = 1 - 4te^{-t}.$$

❶ Chapter 8, Section 11, Problem 5.

Solve the differential equation $y'' + \omega^2 y = f(t)$, $y_0 = y'_0 = 0$, with $f(t)$ given by the functions in Figure 11.4, by the following methods.

- (a) Use the convolution integral, being careful to consider separately the three intervals 0 to t_0 , t_0 to $t_0 + \frac{1}{n}$, and $t_0 + \frac{1}{n}$ to ∞ .
- (b) Write $f_n(t)$ as a difference of unit step functions as in L25, and use L25 to find $L(f_n)$. Expand $\frac{1}{p(p^2 + \omega^2)}$ by partial fractions and use L28 to find $y_n(t)$. Your result should agree with (a).
- (c) Let $n \rightarrow \infty$ and show that your solution in (a) and (b) tends to the same solution (11.5) obtained using the functions of Figure 11.3; that is, either set of functions gives, in the limit, the same solution (11.11) obtained by using the δ function. Note that, when you let $n \rightarrow \infty$, you do not need to consider the interval t_0 to $t_0 + \frac{1}{n}$ since, if $t > t_0$, then for sufficiently large n , $t > t_0 + \frac{1}{n}$.

❷ Chapter 8, Section 11, Problem 7.

Using the δ function method, find the response of $y'' + 2y' + y = \delta(t - t_0)$ to a unit impulse.

❸ Chapter 8, Section 11, Problem 11.

Using the δ function method, find the response of $\frac{d^4 y}{dt^4} - y = \delta(t - t_0)$ to a unit impulse.

Solution:

- ❶ (a) Taking the Laplace transform of each term, substituting the initial conditions, and solving for Y , we get

$$\begin{aligned} p^2 Y + \omega^2 Y &= L(f_n(t)) \\ Y &= \frac{1}{p^2 + \omega^2} L(f_n(t)) \end{aligned}$$

According to L3, the inverse transform of $\frac{1}{p^2 + \omega^2}$ is $\frac{\sin(\omega t)}{\omega}$. Therefore,

$$Y = L\left(\frac{\sin(\omega t)}{\omega}\right) L(f_n(t)).$$

According to L34,

$$y(t) = \int_0^t \frac{\sin[\omega(t-\tau)]}{\omega} f_n(\tau) d\tau$$

$$= \begin{cases} 0 & \text{if } t < t_0 \\ \frac{n}{\omega^2} (1 - \cos[\omega(t - t_0)]) & \text{if } t_0 < t < t_0 + \frac{1}{n} \\ \frac{n}{\omega^2} (\cos[\omega(t - t_0 - \frac{1}{n})] - \cos[\omega(t - t_0)]) & \text{if } t_0 + \frac{1}{n} < t \end{cases}$$

(b) We write $f_n(t)$ as a difference of unit step functions:

$$f_n(t) = n \left(u(t - t_0) - u(t - t_0 - \frac{1}{n}) \right)$$

According to L25,

$$L(f_n(t)) = n \frac{e^{-t_0 p} - e^{-(t_0 + \frac{1}{n})p}}{p}$$

We expand $\frac{1}{p(p^2 + \omega^2)}$ by partial fractions to get

$$\frac{1}{p(p^2 + \omega^2)} = \frac{1}{p} - \frac{p}{p^2 + \omega^2}$$

Putting it all together, we have

$$\begin{aligned} Y &= \frac{1}{p^2 + \omega^2} L(f_n(t)) \\ &= \frac{1}{p^2 + \omega^2} n \frac{e^{-t_0 p} - e^{-(t_0 + \frac{1}{n})p}}{p} \\ &= \frac{n}{p(p^2 + \omega^2)} \left(e^{-t_0 p} - e^{-(t_0 + \frac{1}{n})p} \right) \\ &= n \left(\frac{1}{p} - \frac{p}{p^2 + \omega^2} \right) \left(e^{-t_0 p} - e^{-(t_0 + \frac{1}{n})p} \right) \\ &= n \left(\frac{1}{p} e^{-t_0 p} - \frac{1}{p} e^{-(t_0 + \frac{1}{n})p} - \frac{p}{p^2 + \omega^2} e^{-t_0 p} + \frac{p}{p^2 + \omega^2} e^{-(t_0 + \frac{1}{n})p} \right) \end{aligned}$$

According to L28, the inverse Laplace transform of

$$\begin{aligned} \frac{1}{p} e^{-t_0 p} &\text{ is } u(t - t_0) \\ \frac{1}{p} e^{-(t_0 + \frac{1}{n})p} &\text{ is } \cos(\omega(t - t_0)) u(t - t_0) \\ \frac{p}{p^2 + \omega^2} e^{-t_0 p} &\text{ is } u(t - t_0 - \frac{1}{n}) \\ \frac{p}{p^2 + \omega^2} e^{-(t_0 + \frac{1}{n})p} &\text{ is } \cos(\omega(t - t_0 - \frac{1}{n})) u(t - t_0 - \frac{1}{n}) \end{aligned}$$

Therefore,

$$y_n(t) = \begin{cases} 0 & \text{if } t < t_0 \\ \frac{n}{\omega^2} (1 - \cos[\omega(t - t_0)]) & \text{if } t_0 < t < t_0 + \frac{1}{n} \\ \frac{n}{\omega^2} (\cos[\omega(t - t_0 - \frac{1}{n})] - \cos[\omega(t - t_0)]) & \text{if } t_0 + \frac{1}{n} < t \end{cases}$$

(c) Letting $n \rightarrow \infty$, we see that

$$\lim_{n \rightarrow \infty} y_n(t) = \begin{cases} 0 & \text{if } t < t_0 \\ \frac{\sin(\omega(t-t_0))}{\omega} & \text{if } t > t_0 \end{cases}$$

- ② From Problem 6c, we get the initial conditions $y_0 = y'_0 = 0$. So, taking Laplace transforms, we get

$$(p+1)^2 Y = L(\delta(t-t_0)).$$

According to L6, the Laplace transform of te^{-t} is $\frac{1}{(p+1)^2}$. Therefore,

$$Y = L(te^{-t}) L(\delta(t-t_0)).$$

According to L28,

$$y(t) = (t-t_0)e^{-(t-t_0)}u(t-t_0).$$

- ③ According to 35, taking Laplace transforms yields

$$(p^4-1)Y = L(\delta(t-t_0)).$$

According to Problem 1 of HW12, the Laplace transform of $\frac{\sinh t - \sin t}{2}$ is $\frac{1}{p^4-1}$. Therefore,

$$Y = L\left(\frac{\sinh t - \sin t}{2}\right) L(\delta(t-t_0)).$$

According to L28,

$$y(t) = \frac{\sinh(t-t_0) - \sin(t-t_0)}{2}u(t-t_0).$$

- ❶ Chapter 8, Section 12, Problem 2.

Use (12.6) to solve (12.1) when $f(t) = \sin \omega t$.

- ❷ Chapter 8, Section 12, Problem 6.

For Problem 10.17, show that $G(t, t') = \begin{cases} 0 & 0 < t < t' \\ \frac{\sinh a(t-t')}{a} & t > t' \end{cases}$.

Thus write the solution to 10.17 as an integral similar to (12.6) and evaluate it.

- ❸ Chapter 8, Section 12, Problem 8.

Solve the differential equation $y'' + 2y' + y = f(t)$, $y_0 = y'_0 = 0$, where

$$f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a. \end{cases}$$

As in Problems 6 and 7, find the Green function for the problem and use it in

$$y(t) = \int_0^\infty G(t, t') f(t') dt'. \quad (12.4)$$

Consider the cases $t < a$ and $t > a$ separately.

Solution:

- ❶ We have from (12.6) that

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{\omega} \sin \omega(t - \tau) f(\tau) d\tau \\ &= \frac{1}{\omega} \int_0^t \sin \omega(t - \tau) \sin \omega \tau d\tau \\ &= \frac{1}{\omega} \int_0^t (\sin \omega t \cos \omega \tau - \sin \omega \tau \cos \omega t) \sin \omega \tau d\tau \\ &= \frac{1}{\omega} \int_0^t (\sin \omega t \cos \omega \tau \sin \omega \tau - \sin \omega \tau \cos \omega t \sin \omega \tau) d\tau \\ &= \frac{1}{\omega} \left(\frac{1 - \cos 2\omega t}{4\omega} \sin \omega t - \cos \omega t \left[\frac{t}{2} - \frac{\sin 2\omega t}{4\omega} \right] \right). \end{aligned}$$

② The Green function satisfies

$$\frac{d^2}{dt^2}G(t, t') - a^2G(t, t') = \delta(t - t')$$

with G and $\frac{d}{dt}G$ equal to 0 on $t = 0$. To solve for G , we take the Laplace transform of the differential equation to get

$$(p^2 - a^2)L[G] = L[\delta(t - t')].$$

According to L9, $L[\frac{\sinh at}{a}] = \frac{1}{p^2 - a^2}$. Therefore

$$L[G] = \frac{e^{-t'p}}{p^2 - a^2} = L[\frac{\sinh at}{a}]L[\delta(t - t')].$$

By L28,

$$G(t, t') = \frac{\sinh a(t-t')}{a}u(t - t').$$

Now we solve for $y = y(t)$:

$$\begin{aligned} y(t) &= \int_0^\infty G(t, t')f(t') dt' = \int_0^\infty G(t, t') dt' \\ &= \int_0^t \frac{\sinh a(t-t')}{a} dt' \\ &= -\frac{\cosh a(t-t')}{a^2} \Big|_{t'=0}^t \\ &= \frac{\cosh at - 1}{a^2}. \end{aligned}$$

③ The Green function satisfies

$$\frac{d^2}{dt^2}G(t, t') + 2\frac{d}{dt}G(t, t') + G(t, t') = \delta(t - t')$$

with G and $\frac{d}{dt}G$ equal to 0 on $t = 0$. To solve for G , we take the Laplace transform of the differential equation to get

$$(p^2 + 2p + 1)L[G] = L[\delta(t - t')].$$

Since $\frac{1}{p^2+2p+1} = \frac{1}{(p+1)^2}$, $L[te^{-t}] = \frac{1}{(p+1)^2}$. Therefore

$$L[G] = \frac{e^{-t'p}}{(p+1)^2} = L[te^{-t}]L[\delta(t - t')].$$

By L28,

$$G(t, t') = (t - t')e^{-(t-t')}u(t - t').$$

Now we solve for $y = y(t)$:

$$\begin{aligned} y(t) &= \int_0^\infty G(t, t')f(t') dt' = \int_0^a G(t, t') dt' \\ &= \begin{cases} \int_0^t (t - t')e^{-(t-t')} dt' & \text{if } 0 < t < a \\ \int_0^a (t - t')e^{-(t-t')} dt' & \text{if } 0 < a < t. \end{cases} \end{aligned}$$

Since

$$\int (t - t')e^{-(t-t')} dt' = (t - t' + 1)e^{-(t-t')},$$

if $0 < t < a$, then

$$y(t) = (t - t' + 1)e^{-(t-t')} \Big|_{t'=0}^t = 1 - (t + 1)e^{-t};$$

if $t > a$, then

$$y(t) = (t - t' + 1)e^{-(t-t')} \Big|_{t'=0}^a = (t - a + 1)e^{-(t-a)} - (t + 1)e^{-t}.$$

4.

Use

$$y(x) = -\cos x \int_0^x (\sin \bar{x}) f(\bar{x}) d\bar{x} - \sin x \int_0^{\pi/2} (\cos \bar{x}) f(\bar{x}) d\bar{x} \quad (12.17)$$

to find the solution of

$$y'' + y = f(x) \quad (12.7)$$

with $y(0) = y(\pi/2) = 0$ when the forcing function is given $f(x)$.

- ❶ Chapter 8, Section 12, Problem 11.

$$f(x) = \sin 2x$$

- ❷ Chapter 8, Section 12, Problem 13.

$$f(x) = \begin{cases} x & 0 < x < \pi/4 \\ \pi/2 - x & \pi/4 < x < \pi/2 \end{cases}$$

Solution:

❶

$$\begin{aligned} y(x) &= -\cos x \int_0^x \sin \bar{x} \sin 2\bar{x} d\bar{x} - \sin x \int_0^{\pi/2} \cos \bar{x} \sin 2\bar{x} d\bar{x} \\ &= -\cos x \left(\sin \frac{\bar{x}}{2} - \frac{\sin 3\bar{x}}{6} \right) \Big|_{\bar{x}=0}^x - \sin x \left(\cos \frac{\bar{x}}{2} - \frac{\cos 3\bar{x}}{6} \right) \Big|_{\bar{x}=0}^{\pi/2} \\ &= -\cos x \left(\sin \frac{x}{2} - \frac{\sin 3x}{6} \right) - \frac{\sqrt{2}}{2} \sin x \end{aligned}$$

- ❷ We have two cases: $x < \pi/4$ and $x > \pi/4$.

For $x < \pi/4$,

$$\begin{aligned} y(x) &= -\cos x \int_0^x \bar{x} \sin \bar{x} d\bar{x} - \sin x \left(\int_0^{\pi/4} \bar{x} \cos \bar{x} d\bar{x} + \int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - \bar{x} \right) \cos \bar{x} d\bar{x} \right) \\ &= -\cos x \left(\sin \bar{x} - \bar{x} \cos \bar{x} \right) \Big|_{\bar{x}=0}^x \\ &\quad - \sin x \left(\left(\cos \bar{x} + \bar{x} \sin \bar{x} \right) \Big|_{\bar{x}=0}^{\pi/4} + \frac{\pi}{2} \sin \bar{x} \Big|_{\bar{x}=\pi/4}^{\pi/2} - \left(\cos \bar{x} + \bar{x} \sin \bar{x} \right) \Big|_{\bar{x}=\pi/4}^{\pi/2} \right) \\ &= -\cos x (\sin x - x \cos x) \\ &\quad - \left(\left(\frac{\sqrt{2}}{2} + \frac{\pi}{4} \frac{\sqrt{2}}{2} - 1 \right) + \frac{\pi}{2} \left(1 - \frac{\sqrt{2}}{2} \right) - \left(\frac{\pi}{2} - \frac{\sqrt{2}}{2} - \frac{\pi}{4} \frac{\sqrt{2}}{2} \right) \right) \sin x \\ &= -\cos x (\sin x - x \cos x) - (\sqrt{2} - 1) \sin x \end{aligned}$$

For $x > \pi/4$,

$$\begin{aligned}
y(x) &= -\cos x \left(\int_0^{\pi/4} \bar{x} \sin \bar{x} d\bar{x} + \int_{\pi/4}^x \left(\frac{\pi}{2} - \bar{x} \right) \sin \bar{x} d\bar{x} \right) \\
&\quad - \sin x \left(\int_0^{\pi/4} \bar{x} \cos \bar{x} d\bar{x} + \int_{\pi/4}^{\pi/2} \left(\frac{\pi}{2} - \bar{x} \right) \cos \bar{x} d\bar{x} \right) \\
&= -\cos x \left((\sin \bar{x} - \bar{x} \cos \bar{x}) \Big|_{\bar{x}=0}^{\pi/4} - \frac{\pi}{2} \cos \bar{x} \Big|_{\bar{x}=\pi/4}^x - (\sin \bar{x} - \bar{x} \cos \bar{x}) \Big|_{\bar{x}=\pi/4}^x \right) \\
&\quad - (\sqrt{2} - 1) \sin x \\
&= -\cos x \left(\left(\frac{\sqrt{2}}{2} - \frac{\pi}{4} \frac{\sqrt{2}}{2} \right) - \frac{\pi}{2} (\cos x - \frac{\sqrt{2}}{2}) - \left(\sin x - x \cos x - \left(\frac{\sqrt{2}}{2} - \frac{\pi}{4} \frac{\sqrt{2}}{2} \right) \right) \right) \\
&\quad - (\sqrt{2} - 1) \sin x \\
&= -\cos x \left(\sqrt{2} - \frac{\pi}{2} \cos x - \sin x + x \cos x \right) - (\sqrt{2} - 1) \sin x
\end{aligned}$$

④ 9.2.5.

$$\frac{d}{dx} (2y') = 2y$$

$$\Rightarrow y'' = y \Rightarrow y(x) = c_1 e^x + c_2 e^{-x}$$

$$\text{solve } c_1, c_2 \text{ by } y(x_1) = y_1, \quad y(x_2) = y_2.$$

9.2.9

$$I = \int_{y_1}^{y_2} \left(1 + y \frac{1}{x'(y)} \right)^2 x'(y) dy$$

$$\Rightarrow \frac{\partial}{\partial x'} \left(\left(1 + \frac{y}{x'} \right)^2 x' \right) = 1 - c$$

$$\Rightarrow \frac{\partial}{\partial x'} \left(\left(1 + \frac{y}{x'}\right)^2 x' \right) = 1 - c$$

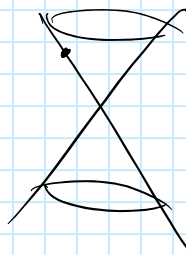
$$\Rightarrow \left(1 + \frac{y}{x'}\right)^2 + 2 \left(1 + \frac{y}{x'}\right) \cdot \left(-\frac{y}{x'^2}\right) = 1 - c$$

$$\Rightarrow 1 - \frac{y^2}{x'^2} = 1 - c \Rightarrow x'^2 = \frac{y^2}{c}$$

$$\Rightarrow x = \frac{1}{2\sqrt{c}} y^2 + \tilde{c}$$

(5) 9.3.17.

$r^2 = z^2$. cylindrical coord.



from (r_1, θ_1, z_1) to (r_2, θ_2, z_2)

① $z_1, z_2 > 0$.

$$\begin{aligned} I &= \int ds = \int \sqrt{dx^2 + dy^2 + dz^2} = \int_{r_1}^{r_2} \sqrt{dr^2 + r^2 d\theta^2 + dr^2} \\ &= \int_{r_1}^{r_2} \sqrt{2 + r^2 \theta'^2} dr \end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial \theta'} \sqrt{2 + r^2 \theta'^2} = c$$

$$\Rightarrow \frac{r^2 \theta'}{\sqrt{2 + r^2 \theta'^2}} = c$$

$$\Rightarrow \theta'^2 = \frac{2c^2}{r^2(r^2 - c^2)}$$

② $z_1 > 0, z_2 < 0$.

need to connect (x_1, y_1, z_1) and $(0, 0, 0)$ and

$(0, 0, 0)$ and (x_2, y_2, z_2)